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ON THE REDUCIBILITY MODULO p OF SIMPLE MODULES

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Abstract

Let $(F; R; k)$ be a splitting p-modular system for the finite group G and let $P \in Syl_p(G)$ fixed. In this paper, we show that a simple kG-module S is the reduction modulo p of an RG-lattice, if and only if S is isomorphic to a direct summand of the induced module from P to G.

1. Introduction

Let G be a finite group, p be a prime divisor of $|G|$, and R be a complete discrete valuation ring with quotient field F of characteristic 0. We assume that the residue field $k = R/J(R)$ has characteristic p, where $J(R)$ denotes the Jacobson radical of R. With this assumption, we refer to the triple $(F; R; k)$ as a splitting p-modular system.

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Recall that the Brauer reduction of a modulo for a natural prime p is defined as follows. If V is an FG-module, then there exists a full RG-lattice $L \subseteq V$. The kG-module $L/J(R)L = U$ is called a reduction of V modulo p. Moreover, in such case, we say also that U is the reduction modulo p of the RG-lattice L.

By Fong-Swan theorem (see [9]), we know that if G is a p-solvable group, then every simple kG-module is the reduction modulo p of an RG-lattice. In our case, firstly, we will study the following problem:

When the simple kG-module S is the reduction modulo p of an RG-lattice L?

2. Preliminary

Let Q be a p-subgroup of the finite group G. Assume that $n = |G : Q|$ and let $D^+ = \{x_1, \dots, x_n\}$ be a full set of representatives in G of the cosets in G/Q. Then $Ind_Q^G(k)$ is isomorphic to kGQ^+ as left kG-module, where $Q^+ = \{ \sum_{x \in D^+} \alpha x \in kG\}$.

Set $X = \{x_i - x_i y,\ y \in Q\}$. We denote the left ideal generated by X in kG by $I_Q(G)$. We claim that

$$\begin{aligned} rank_k(I_Q(G)) &= |G : Q|\,(|Q| - 1) \\ &= |G : P|\,\frac{|P|}{|Q|}\,(|Q| - 1). \end{aligned}$$

Thus, we have

$$kG/I_Q(G) \cong kGQ^+, \qquad (2.1)$$

as k-modules. Now, assume that $Q < Q'$, where Q' is also a p-subgroup of G. Set $X_Q^{Q'} = \{x_i - x_j, x_j = yxy', y \in Q \text{ and } y' \in Q'\}$. Then $kG/I_Q(G)$ contains a left ideal isomorphic to the left ideal generated by $X_Q^{Q'}$. We denote this ideal by $I_Q^{Q'}$. Observe that $rank_k(I_Q^{Q'}) = |G : P| \frac{|P|}{|Q'|}(\frac{|Q'|}{|Q|}|Q| - 1)$. Let us write C_Q by $kG/I_Q(G)$. Thus, we have

$$C_Q/I_Q^{Q'} \cong kGQ'^{+}. \tag{2.2}$$

Remark 2.1. Let G be a finite group with splitting field k of characteristic p, and let S be a simple kG-module. Then P_S denotes the projective cover of S.

Lemma 2.2. *Let G be a finite group with splitting field k of characteristic p, and let S be a simple kG-module. Set $P \in Syl_p(G)$ fixed. Then $P_S^{\dim S}/P_S^{\dim S} I_P(G)$ is a projective kG-module if and only if P_S is a blocks of defect zero.*

Proof. Let $J(G)$ be the Jacobson radical of kG. We to check two cases:

Case I. $J(G) \subseteq I_P(G)$.

Applying the Lemma 2.2, the assertion follows. Conversely, by assumption and applying again the Lemma 2.2, the result follows:

Case II. $J(G) \not\subseteq I_P(G)$.

Assume that $P_S^{\dim S}/P_S^{\dim S} I_P(G) \cong P_S^l$ is a projective kG-module, where l is the multiplicity of P_S as direct summand of $P_S^{\dim S}/P_S^{\dim S} I_P(G)$. We show that P_S is a simple kG-module.

Since $I_P(G)$ is left ideal of kG, we may write

$$I_P(G) = P_{S_1}^{\dim S_1} I_P(G) \oplus \cdots \oplus P_{S_r}^{\dim S_r} I_P(G). \tag{2.3}$$

We claim that $P_S^{\dim S} I_P(G) \cong P_{S_j}^{\dim S_j} I_P(G)$ for some j such that $1 < j \leq r$. Since $P_S^l I_P(G) = 0$, we deduce that $P_S^{\dim S} I_P(G)$ is a projective kG-module, where the multiplicity of P_S is equal to $\dim(S) - l$, i.e., we have

$$P_S^{\dim S} I_P(G) = P_S^{\dim(S)-l}.$$

Therefore, we may assert that $P_S I_P(G)$ is a right indecomposable $I_P(G)$-module such that

$$(P_S I_P(G))^{\dim S} = P_S^{\dim(S)-l}. \tag{2.4}$$

We assume that $\alpha = \dim(P_S I_P(G))$ and $\beta = \dim(P_S)$. According to (2.4), we way write the following equality:

$$\alpha \dim S = \beta(\dim(S) - l). \tag{2.5}$$

From (2.5), it follows that

$$\frac{\alpha}{\dim(S) - l} = \frac{\beta}{\dim S}. \tag{2.6}$$

We now claim that the equality (2.6) is true if and only if $\frac{\alpha}{\dim(S) - l} = \frac{\beta}{\dim S} = 1$. Thus, the following holds $\dim S = \dim P_S$, which is what we need to prove.

Conversely, by assumption, it follows that

$$P_S^{\dim S} I_P(G) = (P_S I_P(G))^{\dim S}, \tag{2.7}$$

where $\dim(P_S I_P(G)) = \dim(S) - l$ with $l = \dim S_{p'}$, being $\dim S_{p'}$ the p'-part of dim S. Thus, we deduce that $P_S^{\dim S} I_P(G) = P_S^{\dim(S)-l}$. So we are done. □

3. Main Results

Proposition 3.3. *Let G be a finite group with splitting field k of characteristic p, and let $P \in Syl_p(G)$ fixed. Then every direct summand of kGP^+ has a radical vertex.*

Proof. Let $N_G(P)$ be the normalizer of P. According to the Green correspondence, all direct summand of $kGP^+ \cong Ind_P^{N_G(P)} Ind_{N_G(P)}^G(k)$ has vertex P or a vertex in $P \bigcap P^g$, $g \in G - N_G(P)$. Assume that U is an indecomposable kG-module with vertex $Q \leq P$, being U a direct summand of kGP^+. We to check two cases:

- **Case 1.** $Q = 1$ or $Q = P$.

The assertion results trivially by assumption.

- **Case 2.** $Q < P$.

In this case, $Q \leq P \bigcap P^g$. Let $N_P(Q)$ be the normalizer of Q in the Sylow p-subgroup P. Since $P \bigcap N_G(Q) = N_P(Q)$ and $P^g \bigcap N_G(Q) = N_P^g(Q)$ are Sylow p-subgroup of $N_G(Q)$, we deduce that $g \in N_G(Q) - N_P(Q)$. We now shows that $N_P(Q)$ is not a normal subgroup of $N_G(Q)$. Let us write $\mathbb{P}$ for $N_P(Q)$. Conversely, we assume that $\mathbb{P}$ is a normal subgroup of $N_G(Q)$. Then, we have

$$N_G(Q) \leq N_G(\mathbb{P}). \tag{3.8}$$

We show that $N_G(\mathbb{P}) \leq N_G(Q)$.

We assume that there is an element $g \in N_G(\mathbb{P})$ such that $Q^g \leq P$ with $Q^g \neq Q$. In such case, we may check that Q^g is a normal subgroup of $\mathbb{P}$, which is immediate. Therefore, we have

$$\mathbb{P} = N_P(Q^g). \tag{3.9}$$

From (3.9), it follows that $Q = Q^g$, which is a contradiction. Thus, we obtain

$$N_G(\mathbb{P}) \leq N_G(Q). \tag{3.10}$$

Combining (3.8) and (3.10), it follows that $N_G(\mathbb{P}) = N_G(Q)$. Now, since $Q \leq \mathbb{P}$, we deduce that $\mathbb{P} = Q$. Hence Q is a radical subgroup of G, which is a vertex of the trivial $N_G(Q)$-module, contradicting $Q < P$. Since $Q = \mathbb{P} \bigcap \mathbb{P}^g$ is the intersection of two Sylow p-subgroups of $N_G(Q)$, we obtain $Q \supseteq O_p(N_G(Q))$. But on the other hand, Q is a normal p-subgroup of $N_G(Q)$, and so is contained in $O_p(N_G(Q))$. Thus, we have equality.

$\square$

Definition 3.4. Let

$$kGP^+ = \bigoplus U,$$

where U is an indecomposable kG-module. If U is a simple kG-module or an indecomposable non-projective kG-module with projective cover P_S, then U is called *G-weight*.

Theorem 3.5. *Let G be a finite group with splitting field k of characteristic p. Then, the number of non-isomorphic G-weights equals the number of conjugacy classes of p-regular elements of G.*

Proof. Since $I_P(G)$ is left ideal of kG, we may write

$$kGP^+ = \bigoplus_{j=1}^{r} M_j^Q, \tag{3.11}$$

where r is the number of conjugacy classes of p-regular elements of G, Q runs over a set of representatives for the conjugacy classes of radical p-subgroups of G, and the M_j^Q are left kG-modules such that

$$M_j^Q \cong P_S^{\dim S}/P_S^{\dim S} I_P(G), \tag{3.12}$$

for some simple kG-module S. Observe that each left kG-module M_j^Q can be decomposed as a direct sum of indecomposable kG-modules, i.e., we may write

$$M_j^Q = \bigoplus_{\gamma=1}^{\mu} U_\gamma, \tag{3.13}$$

where the U_γ are indecomposable kG-modules.

We claim that $M_j^Q/Rad(M_j^Q) \cong \bigoplus S^\mu$, where S is a simple kG-module, i.e., we have $U_1/Rad(U_1) \cong \cdots \cong U_\mu/Rad(U_\mu) \cong S$. We now will prove that in the decomposition (3.13), there is a unique direct summand U_γ, up to isomorphism, which is a G-weight. We to check two cases:

(1) $P \in Syl_p(G)$ and P is a normal subgroup of G.

In such case, P is the unique maximal normal p-subgroup of G. Now, since P acts trivially on every simple kG-module S, we deduce that $J(G) = I_P(G)$. Hence, we have

$$M_j^Q = S_j^{\dim S_j}, \quad j \in \{1, \dots, r\},$$

where S_j is a simple kG-module.

(2) $P \in Syl_p(G)$ and P is not a normal subgroup of G.

Suppose that $M_j^Q = U^l$, being U an indecomposable projective kG-module of multiplicity l. In such case, the result follows by Lemma 2.2.

Therefore, assume that U_γ is direct summand in (3.13), which is an indecomposable non-projective kG-module. Let us now show that P_S is the projective cover of U_γ.

Since $P_S/Rad(P_S) \cong U_\gamma/Rad(U_\gamma) \cong S$, we deduce that there is an epimorphism $P_S \to U_\gamma$, which necessarily is essential by Nakayama's lemma.

We now show that U_γ is unique. Suppose that U_γ and $U_{\gamma'}$ are two G-weights in the decomposition (3.13). Since P_S is projective cover of U_γ and $U_{\gamma'}$, we assert that there are two essential epimorphisms $\theta_1 : P_S \to U_\gamma$ and $\theta_2 : P_S \to U_{\gamma'}$. We define the homomorphism $f : U_\gamma \to U_{\gamma'}$ given by $f(\theta_1(a)) = \theta_2(a)$, $a \in P_S$. Applying again Nakayama's lemma, we deduce that f is an isomorphism. Therefore, the following holds $U_\gamma \cong U_{\gamma'}$, which is what we need to prove.

□

Let G be a finite group and let $\mathbb{C}^{p-reg(G)}$ be the vector space of class functions on the p-regular elements of G. Then, we may define a Hermitian bilinear form on this space by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{p-regular\ g \in G} \overline{\phi(g)}\psi(g).$$

Now, if P and U are finite-dimensional kG-modules with P projective, then

$$\dim Hom_{kG}(P, U) = \langle \phi_P, \phi_U \rangle. \tag{3.14}$$

Let G be a finite group and k be a splitting field for G of characteristic p. Let $S_1, \dots, S_r$ be a complete list of non-isomorphic simple kG-modules. Then, the Brauer characters $\phi_{S_1}, \dots, \phi_{S_r}$ of the simple modules form a basis for $\mathbb{C}^{p-reg(G)}$.

Lemma 3.6. *Let G be a finite group and k be a splitting field for G. Let $U_1, \dots, U_r$ be a complete list of non-isomorphic G-weights, with projective covers $P_{S_1}, \dots, P_{S_r}$. Then, the Brauer characters $\phi_{U_1}, \dots, \phi_{U_r}$ of the G-weights form a basis in the space $\mathbb{C}^{p-reg(G)}$ of class functions on the p-regular elements of G.*

Proof. Everything follows from the formula:

$$\tau = \langle \phi_{P_{S_i}}, \phi_{U_j} \rangle = \begin{cases} \tau = 0, & \text{if } i \neq j; \\ \tau = 1, & \text{if } i = j \text{ and } U_j \cong S_i; \\ \tau > 1, & \text{if } i = j \text{ and } U_j \ncong S_i, \end{cases}$$

and the fact that the number of non-isomorphic G-weights modules equals the number of p-regular conjugacy classes of G. Thus if $\sum_{i=1}^{r} \lambda_i \phi_{U_i} = 0$, we have $\langle \phi_{P_{S_i}}, \phi_{U_i} \rangle \lambda_i = 0$, so $\lambda_i = 0$, which shows that the are independent, and hence form a basis. □

Theorem 3.7. *Let $(F; R; k)$ be a splitting p-modular system for the finite group G. The simple kG-module S is the reduction modulo p of an RG-lattice if and only if S is a G-weight.*

Proof. Let S be a simple kG-module with projective cover P_S, and let U_i be a G-weight such that $U_i/Rad(U_i) \cong S$. Assume that S is the reduction modulo p of an RG-lattice. According to the Lemma 3.6, we may write

$$\sum_{i=1}^{r} \lambda_i \phi_{U_i} = \phi_S. \tag{3.15}$$

From (3.15), we may write

$$\langle \phi_S, \phi_{U_i} \rangle \lambda_i = \langle \phi_S, \phi_S \rangle. \tag{3.16}$$

Since S and U_i are liftable to one RG-lattice, and S is the radical quotient of U_i, it follows that $\langle \phi_S, \phi_{U_i} \rangle = \langle \phi_S, \phi_S \rangle$, so $\lambda_i = 1$.

Conversely, since kGP^+ is the reduction modulo p of the RG-lattice RGP^+ the result follows. □

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